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FORCED VERTICAL VIBRATION OF FOUR RIGID STRIPS ON A SEMI-INFINITE ELASTIC SOLID

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Abstract—In this paper, the problem of two-dimensional oscillations of four rigid strips, situated on a homogeneous isotropic semi-infinite elastic solid and forced by a specified normal component of the displacement has been considered. The mixed boundary value problem of determining the unknown stress distribution just below the strips and vertical displacement outside the strips has been converted to the determination of the solution of quadruple integral equations by the use of Fourier transform. An iterative solution of these integral equations valid for low frequency has been found by the application of the finite Hilbert transform. The normal stress just below the strips and the vertical displacement away from the strips have been obtained. Finally, graphs are presented which illustrate the salient features of the displacement and stress intensity factors at the edges of the strips. © 1997, Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The problem of the effect of vibrating source in different forms on the surface of an elastic medium have aroused attention in view of their application in seismology and geophysics. Reissner (1937), and later Millar and Pursey (1954), treated the case of a uniform vibrating pressure distribution applied to a circular region on the surface of an elastic half-space. Analytical treatment of the dynamical response of footings and solid–structure interaction are usually available in the literature only for circular and elliptical footings, and infinite strip loadings. Such results are important in view of their application in the design of foundations for machinery and buildings, and also in the study of the vibration of dams and large structures subjected to earthquakes. The problem of circular punch has been solved analytically by Awojobi and Grootenhuis (1965), Robertson (1966), Gladwell (1968) and others. Roy (1986) considered the dynamic response of an elliptical footing in frictionless contact with a homogeneous elastic half-space. Karasudhi *et al.* (1968) obtained a low frequency solution for the vertical, horizontal and rocking vibration of an infinite strip on a semi-infinite elastic medium. Wickham (1977) worked out in detail the problem of forced two-dimensional oscillation of a rigid strip in smooth contact with a semi-infinite elastic medium. Recently, Mandal and Ghosh (1992) treated the problem of forced vertical vibration of two rigid strips on a semi-infinite elastic medium.

To improve the dynamic models of buildings and other structures, it will be fruitful to have analytic results for foundations of a more complicated nature. In what follows, the problem of vertical vibration of four rigid strips in smooth contact with a semi-infinite elastic medium has been considered. The problem is also important in view of its application in the study of the vibration of an elastic medium caused by running wheels on a railway track. The resulting mixed boundary value problem has been reduced to the solution of

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quadruple integral equations, which have further been reduced to the solution of integral-differential equations. Finally, an iterative solution valid for low frequency has been obtained.

From the solution of the integral equations, the stress just below the strips and also the vertical displacement at points outside the strips on the free surface have been found. The effects of stress intensity factors at the edges of the strips and vertical displacement outside the strips have been shown by means of graphs.

2. FORMULATION OF THE PROBLEM

Consider the normal vibration of frequency ω of four rigid strips having smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty < X < \infty$, $Y \geqslant 0$, $-\infty < Z < \infty$. It is assumed that the motion is forced by prescribed displacement distribution $v_0 e^{-i\omega t}$ normal to the four infinite strips located in the region $d_1 \leqslant |X| \leqslant d_2$, $d_3 \leqslant |X| \leqslant d$, $Y = 0$, $|Z| < \infty$, where v_0 is a constant.

Normalizing all the lengths with respect to d and putting $X/d = x$, $Y/d = y$, $Z/d = z$, $d_1/d = a$, $d_2/d = b$, $d_3/d = c$, one finds that the rigid strips are defined by $a \leqslant |x| \leqslant b$, $c \leqslant |x| \leqslant 1$, $y = 0$, $|z| < \infty$ (Fig. 1).

With the time factor $e^{-i\omega t}$ suppressed throughout the analysis, the displacement components can be written as

$$u(x,y) = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}; \quad v(x,y) = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}; \quad w(x,y) = 0 \tag{1}$$

where the displacement potentials $\phi(x,y)$ and $\psi(x,y)$ satisfy the Helmholtz equations

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + m_1^2 \phi = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + m_2^2 \psi = 0 \tag{2}$$

in which

$$m_1^2 = \frac{\omega^2 d^2}{c_1^2} \quad \text{and} \quad m_2^2 = \frac{\omega^2 d^2}{c_2^2}.$$

In terms of ϕ and ψ the stress components are

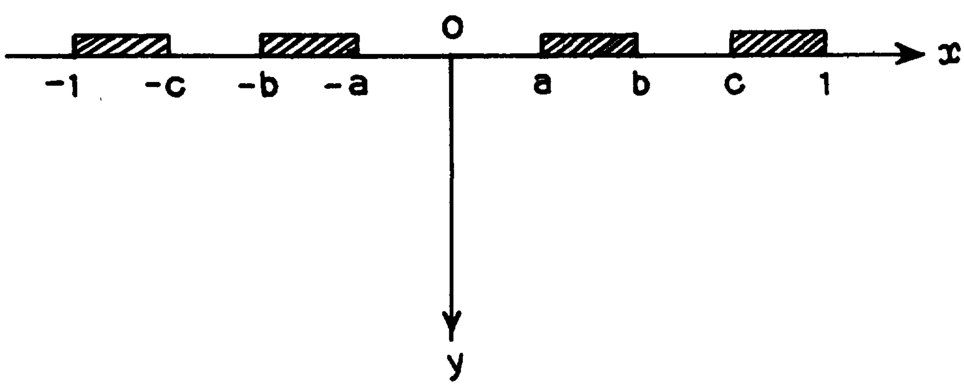

Fig. 1. Geometry of the problem.

$$\tau_{xy} = \mu\left\{2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right\}$$

$$\tau_{yy} = -\mu\left\{\left(m_2^2+2\frac{\partial^2}{\partial x^2}\right)\phi - 2\frac{\partial^2\psi}{\partial x\partial y}\right\}$$

$$\tau_{yz} = 0. \tag{3}$$

The boundary conditions are

$$v(x,0) = v_0, \quad x\in I_2, I_4 \tag{4}$$

$$\tau_{yy}(x,0) = 0, \quad x\in I_1, I_3, I_5 \tag{5}$$

$$\tau_{xy}(x,0) = 0, \quad -\infty < x < \infty \tag{6}$$

where $I_1 = (0,a), I_2 = (a,b), I_3 = (b,c), I_4 = (c,1), I_5 = (1,\infty)$. The solution of the Helmholtz equation (2) can be written as

$$\phi = 2\int_0^\infty A(\xi)\cos\xi x e^{-\gamma_1 y}\,d\xi$$

$$\psi = 2\int_0^\infty B(\xi)\sin\xi x e^{-\gamma_2 y}\,d\xi \tag{7}$$

where

$$\gamma_j = \begin{cases}(\xi^2-m_j^2)^{1/2}, & |\xi| \geqslant m_j\\ -i(m_j^2-\xi^2)^{1/2}, & |\xi| \leqslant m_j\end{cases}, \quad j = 1,2$$

and $A(\xi)$ and $B(\xi)$ are unknown functions to be determined from the boundary conditions.

By using the boundary condition (6), it can be shown that

$$B(\xi) = \frac{2\gamma_1\xi}{\xi^2+\gamma_2^2}A(\xi). \tag{8}$$

Now the displacement component v and stress τ_{yy} become

$$v(x,y) = 2\int_0^\infty\left[\frac{2\xi^2}{2\xi^2-m_2^2}e^{-\gamma_2 y} - e^{-\gamma_1 y}\right]A(\xi)\cos\xi x\,d\xi \tag{9}$$

$$\tau_{yy}(x,y) = -2\mu\int_0^\infty\left[(m_2^2-2\xi^2)e^{-\gamma_1 y} + \frac{2\xi^2\gamma_1\gamma_2}{2\xi^2-m_2^2}e^{-\gamma_2 y}\right]A(\xi)\cos\xi x\,d\xi. \tag{10}$$

From the boundary conditions (4) and (5) we get the following set of integral equations in $P(\xi)$:

$$\int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}P(\xi)\cos\xi x\,d\xi = \frac{1}{2}v_0, \quad x\in I_2, I_4 \tag{11}$$

and

$$\int_0^\infty P(\xi)\cos\xi x\,\mathrm{d}\xi = 0, \quad x\in I_1, I_2, I_5 \tag{12}$$

where

$$P(\xi) = \frac{(2\xi^2-m_2^2)^2+4\xi^2\gamma_1\gamma_2}{(2\xi^2-m_2^2)}A(\xi).$$

3. SOLUTION OF THE PROBLEM

We consider the solution of the integral equations (11) and (12) in the form

$$P(\xi) = \int_a^b tf(t^2)\cos\xi t\,\mathrm{d}t+\int_c^1 ug(u^2)\cos\xi u\,\mathrm{d}u \tag{13}$$

where $f(t^2)$ and $g(u^2)$ are unknown functions to be determined.

By the choice of $P(\xi)$ given by eqn (13) the relation (12) is satisfied automatically and eqn (11) becomes

$$\int_a^b tf(t^2)\,\mathrm{d}t\int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\cos\xi x\cos\xi t\,\mathrm{d}\xi$$
$$+\int_c^1 ug(u^2)\,\mathrm{d}u\int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\cos\xi x\cos\xi u\,\mathrm{d}\xi = \frac{v_0}{2}, \quad x\in I_2, I_4 \tag{14}$$

using the relation

$$\frac{\sin\xi x\sin\xi t}{\xi^2} = \int_0^x\int_0^t \frac{wvJ_0(\xi w)J_0(\xi v)\,\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$

the above equation is converted to the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b tf(t^2)\,\mathrm{d}t\frac{\partial}{\partial t}\int_0^x\int_0^t \frac{wvL_1(v,w)\,\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}+$$
$$+\frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 ug(u^2)\,\mathrm{d}u\frac{\partial}{\partial u}\int_0^x\int_0^u \frac{wvL_1(v,w)\,\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}} = \frac{v_0}{2}, \quad x\in I_2, I_4 \tag{15}$$

where

$$L_1(v,w) = \int_0^\infty \frac{\gamma_1 m_2^2}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}J_0(\xi w)J_0(\xi v)\,\mathrm{d}\xi. \tag{16}$$

By a simple contour integration technique used by Ghosh and Ghosh (1985), $L_1(v, w)$ can be written as

$$L_1(v,w) = -\mathrm{i}\tau^2\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\mathrm{d}\eta$$
$$-4\mathrm{i}\tau^2\int_0^\tau \frac{\eta^2(\eta^2-1)(\tau^2-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\mathrm{d}\eta$$

$$+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \quad w>v$$

$$=\frac{-i\tau^2}{16(1-\tau^2)}\left[\sum_{j=0}^{2}P_j\int_0^1\frac{(1-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{\eta^2-\tau_j^2}\mathrm{d}\eta\right.$$

$$\left.+\sum_{j=0}^{2}S_j\int_0^{\tau}\frac{(\tau^2-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{\eta^2-\tau_j^2}\mathrm{d}\eta\right]$$

$$+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \quad w>v \qquad (17)$$

where

$$\tau=\frac{m_2}{m_1}=\frac{c_1}{c_2}, \quad Q_0(\eta)=(2\eta^2-\tau^2)^2-4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}$$

and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta)=0$. τ_1, τ_2 are the roots of the equation

$$(2\eta^2-\tau^2)^2+4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}=0.$$

$Q_0'(\eta)$ denotes the derivative of $Q_0(\eta)$ with respect to η and

$$P_j=\frac{(2\tau_j^2-\tau^2)}{\prod_i(\tau_j^2-\tau_i^2)}, \quad S_j=\frac{4\tau_j^2(\tau_j^2-1)}{\prod_i(\tau_j^2-\tau_i^2)}, \quad i,j=0,1,2 \quad \text{and} \quad i\neq j.$$

The corresponding expression for $L_1(v,w)$ for $w<v$ follows from eqn (17) by interchanging w and v. For a Poisson ratio $\sigma=\frac{1}{4}$, the values of τ, τ_0, τ_1 and τ_2 are given by

$$\tau^2=\frac{2(1-\sigma)}{(1-2\sigma)}=3, \quad \tau_0^2=\frac{3}{(0.9194)^2}, \quad \tau_1^2=\frac{3}{(2+2\sqrt{3})} \quad \text{and} \quad \tau_2^2=\frac{3}{4}.$$

Hence, in this case $\tau_2<\tau_1<1<\tau<\tau_0$.

By using the series expansions of J_0 and $H_0^{(1)}$, and evaluating the integrals arising in eqn (17), we obtain, after some algebraic manipulation,

$$L_1(v,w)=\frac{2}{\pi}\tau^2\left[\left(\gamma+\log\frac{m_1w}{2}-\frac{\pi i}{2}\right)M+N-\frac{P}{4}(w^2+v^2)m_1^2\log m_1\right]+O(m_1^2) \quad w>v.$$

$$=\frac{2}{\pi}\tau^2\left[\left(\gamma+\log\frac{m_1v}{2}-\frac{\pi i}{2}\right)M+N-\frac{P}{4}(w^2+v^2)m_1^2\log m_1\right]+O(m_1^2) \quad w<v, \qquad (18)$$

where $\gamma=0.5772157\ldots$ is Euler's constant,

$$M=-\frac{\pi}{4(1-\tau^2)} \qquad (19)$$

$$N = \frac{\pi}{32(1-\tau^2)}\left[4\log\frac{4}{\tau} + \sum_{j=1}^{2} P_j \frac{\sqrt{(1-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(1-\tau_j^2)}}{\tau_j} - P_0\frac{\sqrt{(\tau_0^2-1)}}{\tau_0}\right.$$
$$\times\log\{\tau_0+\sqrt{(\tau_0^2-1)}\} + \sum_{j=1}^{2} S_j\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j}$$
$$\left. - S_0\frac{\sqrt{(\tau_0^2-\tau^2)}}{\tau_0}\log\left\{\frac{\tau_0+\sqrt{(\tau_0^2-\tau^2)}}{\tau}\right\}\right], \quad (20)$$

$$P = \frac{\pi}{32(1-\tau^2)}\left[\sum_{j=0}^{2} P_j\left(\frac{1}{2}-\tau_j^2\right) + \sum_{j=0}^{2} S_j\left(\frac{\tau^2}{2}-\tau_j^2\right)\right]. \quad (21)$$

Next, differentiating both sides of relation (14) with respect to x, we obtain

$$\int_a^b tf(t^2)\,\mathrm{d}t\int_0^\infty \frac{\gamma_1 m_2^2\xi}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\sin\xi x\sin\xi t\,\mathrm{d}\xi$$
$$+\int_c^1 ug(u^2)\,\mathrm{d}u\int_0^\infty \frac{\gamma_1 m_2^2\xi}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\sin\xi x\sin\xi u\,\mathrm{d}\xi = 0, \quad x\in I_2, I_4.$$

Following a similar procedure as for deriving eqn (15), we get

$$x\int_a^b \frac{tf(t^2)}{x^2-t^2}\mathrm{d}t + x\int_c^1 \frac{ug(u^2)}{x^2-u^2}\mathrm{d}u = \int_a^b tf(t^2)\,\mathrm{d}t\frac{\partial}{\partial t}\int_0^x\int_0^t \frac{wvL_2(v,w)\,\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}$$
$$+\int_c^1 ug(u^2)\,\mathrm{d}u\frac{\partial}{\partial u}\int_0^x\int_0^u \frac{wvL_2(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}}, \quad x\in I_2, I_4 \quad (22)$$

where

$$L_2(v,w) = \int_0^\infty\left[\xi - \frac{2\gamma_1\xi^2(m_1^2-m_2^2)}{(2\xi^2-m_2^2)^2-4\xi^2\gamma_1\gamma_2}\right]J_0(\xi w)J_0(\xi v)\,\mathrm{d}\xi. \quad (23)$$

For small values of m_1 and m_2 such that $m_1 = O(m_2)$, one can use the contour integration technique mentioned above and obtain

$$L_2(v,w) = 2\mathrm{i}m_1^2(1-\tau^2)\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\eta^2 H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\mathrm{d}\eta$$
$$+4\mathrm{i}m_1^2(1-\tau^2)\int_0^\tau -\frac{2\eta^4(\eta^2-1)(\tau^2-\eta^2)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\mathrm{d}\eta$$
$$-2\pi\mathrm{i}m_1^2(1-\tau^2)\left[\frac{\eta^2(\eta^2-1)^{1/2}H_0^{(1)}(m_1\eta w)J_0(m_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \quad w>v \quad (24)$$

By a process similar to the one which led to eqn (18), eqn (24) can be written as

$$L_2(v,w) = -\frac{4P}{\pi}(1-\tau^2)m_1^2\log m_1 + O(m_1^2) \quad (25)$$

where P is given by eqn (21).

Now examining relations (15) and (18), we assume the expressions of the functions $f(t^2)$ and $g(u^2)$ as

$$f(t^2) = f_0(t^2)+f_1(t^2)m_1^2\log m_1+O(m_1^2)$$
$$g(u^2) = g_0(u^2)+g_1(u^2)m_1^2\log m_1+O(m_1^2). \tag{26}$$

Putting the above expressions of $f(t^2)$ and $g(u^2)$, and the value of $L_2(v, w)$ given by eqn (25) in eqn (22) and equating the coefficients of like powers of m_1 we obtain

$$\int_a^b \frac{tf_0(t^2)}{x^2-t^2}\,dt+\int_c^1 \frac{ug_0(u^2)}{x^2-u^2}\,du = 0, \quad x\in I_2, I_4 \tag{27}$$

and

$$\int_a^b \frac{tf_1(t^2)}{x^2-t^2}\,dt+\int_c^1 \frac{ug_1(u^2)}{x^2-u^2}\,du = -\frac{4}{\pi}P(1-\tau^2)\left[\int_a^b tf_0(t^2)\,dt+\int_c^1 ug_0(u^2)\,du\right], \quad x\in I_2, I_4. \tag{28}$$

Following Srivastava and Lowengrub (1970), the solutions of the above integral equations (27) can be obtained as

$$f_0(t^2) = D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2}\frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$
$$-D_2\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\frac{1}{\sqrt{(1-t^2)(c^2-t^2)}}, \quad t\in I_2 \tag{29}$$

and

$$g_0(u^2) = D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2}\frac{1}{\sqrt{(u^2-a^2)(u^2-b^2)}}$$
$$+D_2\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}\frac{1}{\sqrt{(u^2-c^2)(1-u^2)}}, \quad u\in I_4 \tag{30}$$

where D_1 and D_2 are constants which can be calculated as follows:

We substitute the value of $L_1(v, w)$ from eqn (18), as well as the expansions of $f(t^2)$ and $g(u^2)$ obtained from eqns (26), (29) and (30) up to $O(m_1^2\log m_1)$ in eqn (15). When the coefficients of like powers of m_1 from both sides of the resulting equation are equated, after some algebraic manipulation we get the following

$$D_1 = \frac{\pi v_0}{4\tau^2}\frac{(X_2-X_1)}{(X_1X_4-X_2X_3)}; \quad D_2 = \frac{\pi v_0}{4\tau^2}\frac{(X_1-X_3)}{(X_1X_4-X_2X_3)} \tag{31}$$

where

$$X_1 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left[\left\{\left(\gamma+\log\frac{m_1}{2}-\frac{\pi i}{2}\right)M+N\right\}(J_1+J_3)+\frac{1}{2}MJ_1\log(b^2-a^2)+MJ_5\right] \tag{32}$$

$$X_2 = \left\{\left(\gamma+\log\frac{m_1}{2}-\frac{\pi i}{2}\right)M+N\right\}(J_4-J_2)-\frac{1}{2}MJ_2\log(b^2-a^2)+MJ_6 \tag{33}$$

$$X_3 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left[\left\{\left(\gamma+\log\frac{m_1}{2}-\frac{\pi i}{2}\right)M+N\right\}(J_1+J_3)+\frac{1}{2}MJ_3\log(1-c^2)+MJ_7\right] \tag{34}$$

$$X_4 = \left\{\left(\gamma+\log\frac{m_1}{2}-\frac{\pi i}{2}\right)M+N\right\}(J_4-J_2)+\frac{1}{2}MJ_4\log(1-c^2)-MJ_8$$

$$J_1 = \int_a^b\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2}\frac{t\,\mathrm{d}t}{\sqrt{(t^2-a^2)(b^2-t^2)}};\quad J_2 = \int_a^b\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\frac{t\,\mathrm{d}t}{\sqrt{(1-t^2)(c^2-t^2)}}$$

$$J_3 = \int_c^1\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2}\frac{u\,\mathrm{d}u}{\sqrt{(u^2-a^2)(u^2-b^2)}};\quad J_4 = \int_c^1\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}\frac{u\,\mathrm{d}u}{\sqrt{(u^2-c^2)(1-u^2)}}$$

$$J_5 = \int_c^1\frac{u\log(\sqrt{u^2-b^2}+\sqrt{u^2-a^2})}{\sqrt{(u^2-a^2)(u^2-b^2)}}\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2}\mathrm{d}u$$

$$J_6 = \int_c^1\frac{u\log(\sqrt{u^2-b^2}+\sqrt{u^2-a^2})}{\sqrt{(1-u^2)(u^2-c^2)}}\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}\mathrm{d}u$$

$$J_7 = \int_a^b\frac{t\log(\sqrt{c^2-t^2}+\sqrt{1-t^2})}{\sqrt{(t^2-a^2)(b^2-t^2)}}\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2}\mathrm{d}t$$

$$J_8 = \int_a^b\frac{t\log(\sqrt{c^2-t^2}+\sqrt{1-t^2})}{\sqrt{(1-t^2)(c^2-t^2)}}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\mathrm{d}t. \tag{35}$$

4. STRESS INTENSITY FACTORS AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ on the plane $y = 0$ can be found from the relations (10), (13), (26), (29) and (30) as

$$\begin{aligned}\tau_{yy}(x,0) &= \frac{\pi\mu x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\left[D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{c^2-x^2}{1-x^2}\right)^{1/2}\right.\\ &\quad\left.-D_2\frac{(x^2-a^2)}{\sqrt{(1-x^2)(c^2-x^2)}}\right]+O(m_1^2\log m_1),\quad x\in I_2\\ &= \frac{\pi\mu x}{\sqrt{(x^2-c^2)(1-x^2)}}\left[D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\frac{(x^2-c^2)}{\sqrt{(x^2-a^2)(x^2-b^2)}}\right.\\ &\quad\left.+D_2\left(\frac{x^2-a^2}{x^2-b^2}\right)^{1/2}\right]+O(m_1^2\log m_1),\quad x\in I_4.\end{aligned} \tag{36}$$

Defining the stress intensity factors at the edges of the strips by the relations

$$K_a = \underset{x\to a+}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{x-a}}{\pi\mu v_0}\right|;\quad K_b = \underset{x\to b-}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{b-x}}{\pi\mu v_0}\right|$$

$$K_c = \underset{x\to c+}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{x-c}}{\pi\mu v_0}\right|;\quad K_1 = \underset{x\to 1-}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{1-x}}{\pi\mu v_0}\right|.$$

We get

$$K_a = \left|\frac{\sqrt{a}D_1/v_0}{\sqrt{2(b^2-a^2)}}\right| \tag{37}$$

$$K_b = \left|\frac{\sqrt{b}}{\sqrt{2(b^2-a^2)}}\left\{\frac{D_1}{v_0}\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{c^2-b^2}{1-b^2}\right)^{1/2} - \frac{D_2}{v_0}\frac{(b^2-a^2)}{\sqrt{(1-b^2)(c^2-b^2)}}\right\}\right| \tag{38}$$

$$K_c = \left|\frac{\sqrt{c}}{\sqrt{2(1-c^2)}}\frac{D_2}{v_0}\left(\frac{c^2-a^2}{c^2-b^2}\right)^{1/2}\right| \tag{39}$$

$$K_1 = \left|\frac{1}{\sqrt{2(1-c^2)}}\left\{\frac{(1-c^2)D_1}{\sqrt{(c^2-a^2)(1-b^2)}} + \left(\frac{1-a^2}{1-b^2}\right)^{1/2} D_2\right\}\right|. \tag{40}$$

The vertical displacement $v(x, y)$ on the plane $y = 0$ can be obtained from eqns (9), (13), (26), (29) and (30) as

$$v(x,0) = \frac{4\tau^2}{\pi}\left[\left\{\left(\gamma + \log m_1 - \frac{\pi i}{2}\right)M + N\right\}\left\{D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}(J_1+J_3) + D_2(J_4-J_2)\right\} + \frac{M}{2}\left\{(J_9+J_{11})\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} D_1 + D_2(J_{12}-J_{10})\right\}\right] \quad x \in I_1, I_3, I_5 \tag{41}$$

where

$$J_9 = \int_a^b \frac{t\log|t^2-x^2|}{\sqrt{(t^2-a^2)(b^2-t^2)}}\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2} \mathrm{d}t$$

$$J_{10} = \int_a^b \frac{t\log|t^2-x^2|}{\sqrt{(1-t^2)(c^2-t^2)}}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2} \mathrm{d}t$$

$$J_{11} = \int_c^1 \frac{u\log|u^2-x^2|}{\sqrt{(u^2-a^2)(u^2-b^2)}}\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2} \mathrm{d}u$$

$$J_{12} = \int_c^1 \frac{u\log|u^2-x^2|}{\sqrt{(u^2-c^2)(1-u^2)}}\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2} \mathrm{d}u.$$

5. NUMERICAL RESULTS AND DISCUSSION

The stress intensity factors (SIF) K_a, K_b, K_c and K_1 at the edges of the strips and vertical displacement $|v(x, 0)/v_0|$ near the rigid strips have been plotted against dimensionless frequency m_1 and distance x, respectively, for a Poisson solid ($\tau^2 = 3$).

It is found that whatever the lengths of the strips are, SIFs at the four edges of the strips increase with an increase in the value of m_1 ($0.1 \leqslant m_1 \leqslant 0.6$).

From the graphs, it may be further noted that with a decrease in the length of the inner strip, which might be induced either by increasing "a" or by decreasing "b" the SIFs gradually increase (Figs 2–9).

Also, a decrease in the value of the length of the outer strip, which might be induced by increasing the value of c, causes an increase in the values of the SIFs (Figs 10–13), from which an interesting conclusion might be drawn: i.e. that the presence of the outer strip suppresses the SIFs at both the edges of the inner strip and the presence of the inner strip suppresses the SIFs at both the edges of the outer strip.

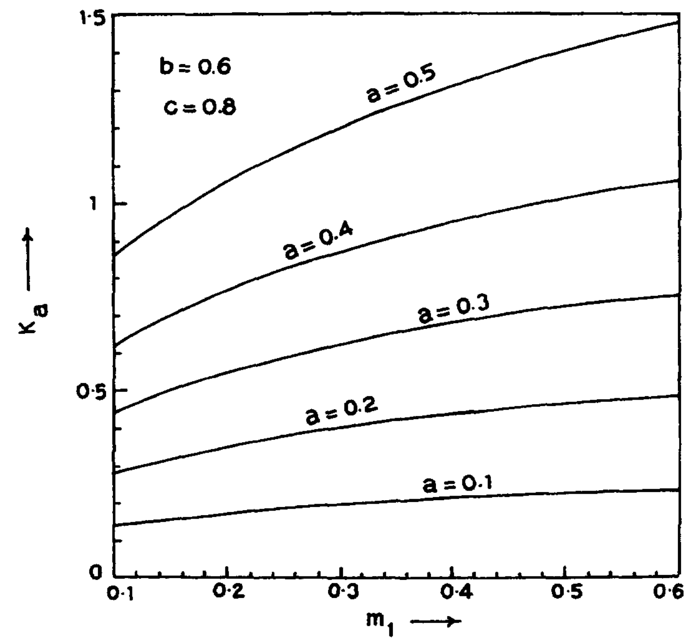

Fig. 2. Stress intensity factor K_a vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

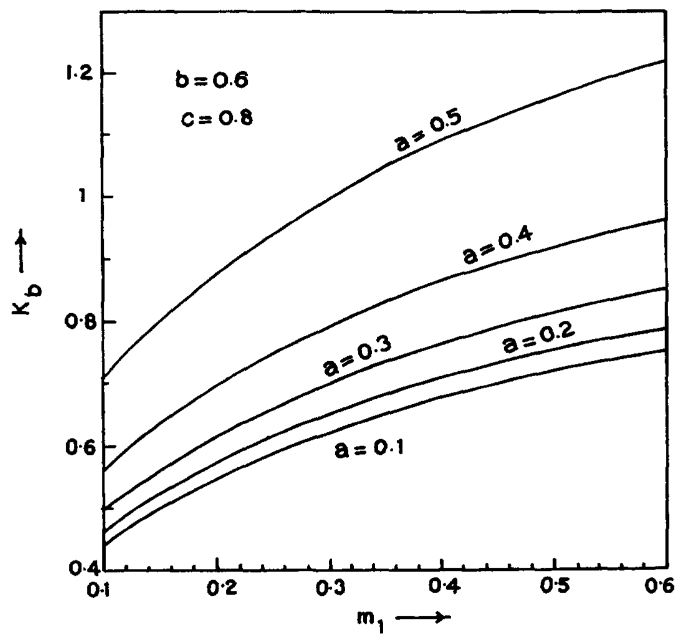

Fig. 3. Stress intensity factor K_b vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

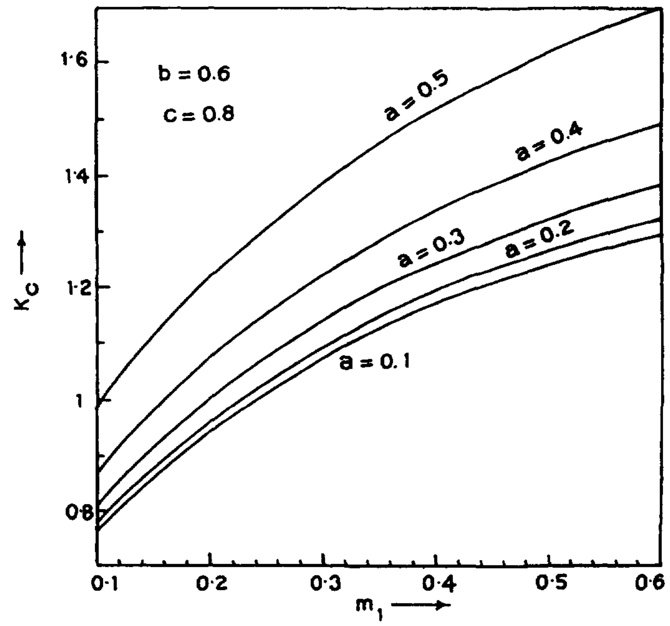

Fig. 4. Stress intensity factor K_c vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

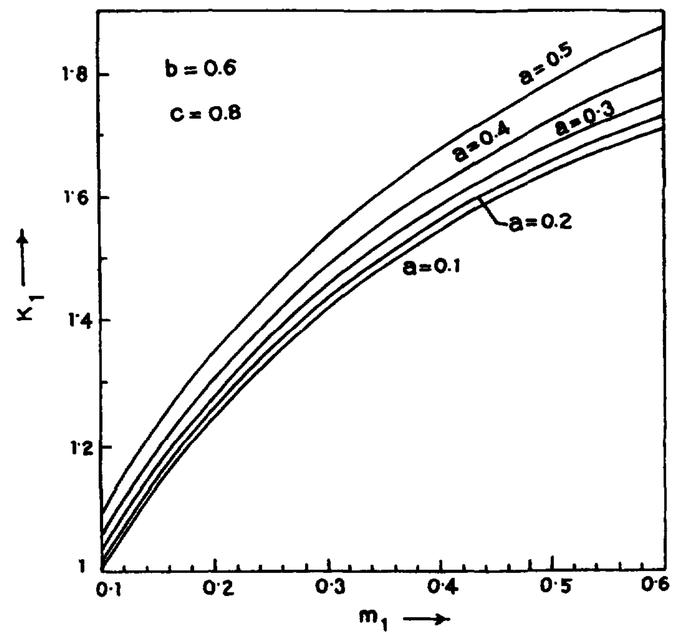

Fig. 5. Stress intensity factor K_1 vs dimensionless frequency m_1 for $b = 0.6$, $c = 0.8$ and for different values of a.

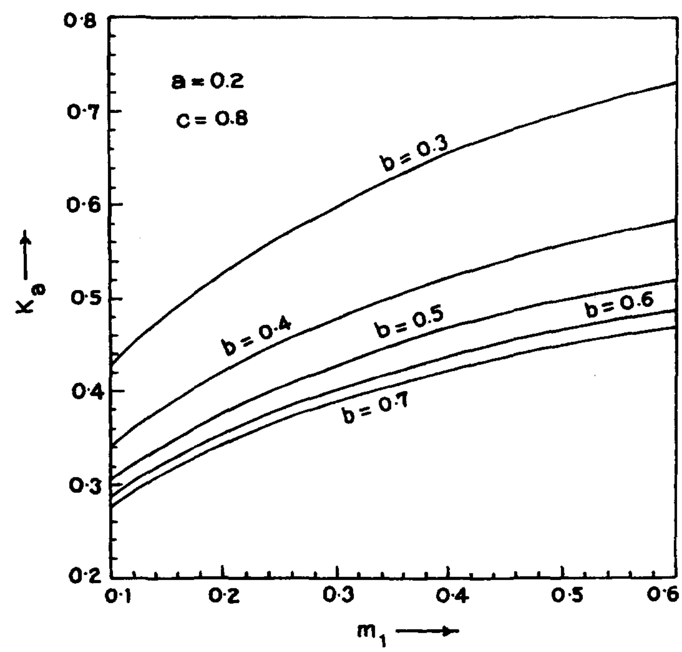

Fig. 6. Stress intensity factor K_a vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

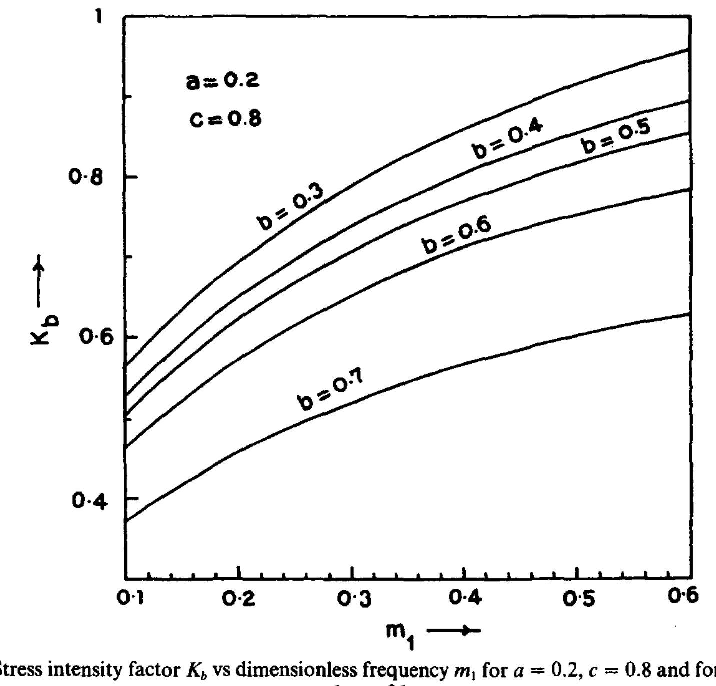

Fig. 7. Stress intensity factor K_b vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

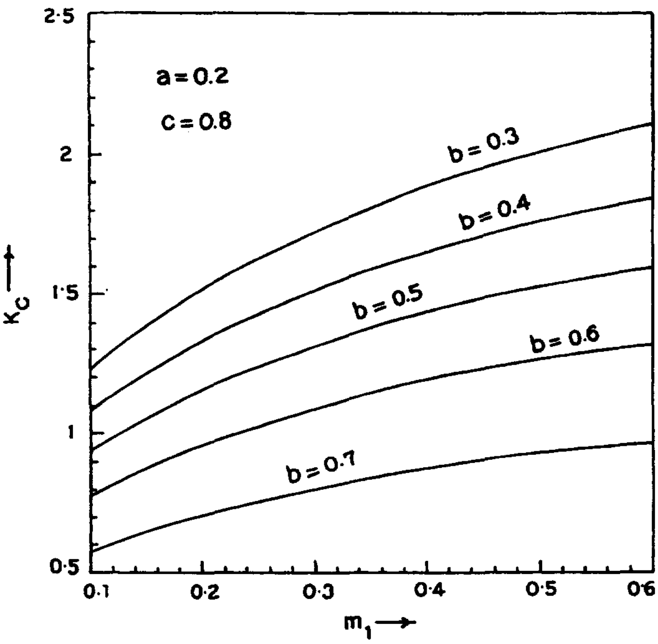

Fig. 8. Stress intensity factor K_c vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

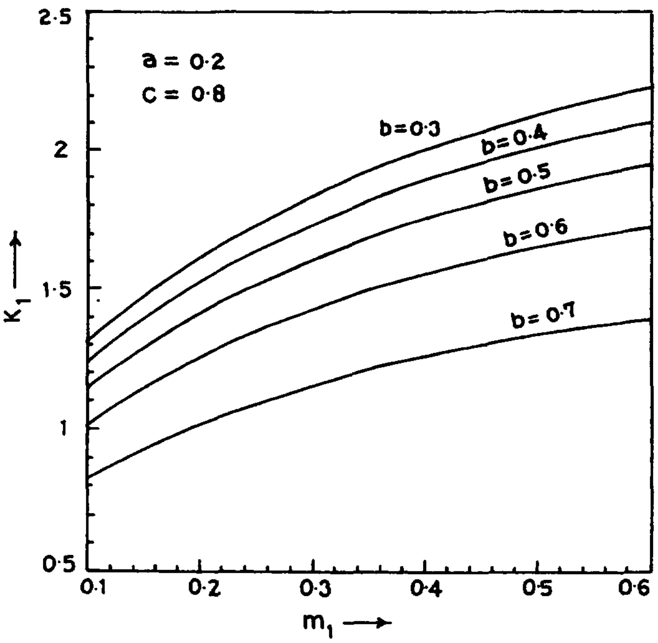

Fig. 9. Stress intensity factor K_1 vs dimensionless frequency m_1 for $a = 0.2$, $c = 0.8$ and for different values of b.

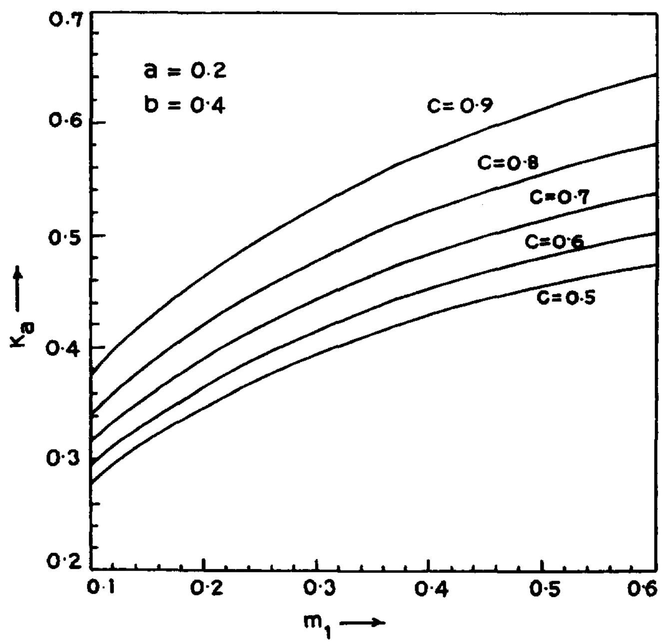

Fig. 10. Stress intensity factor K_a vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

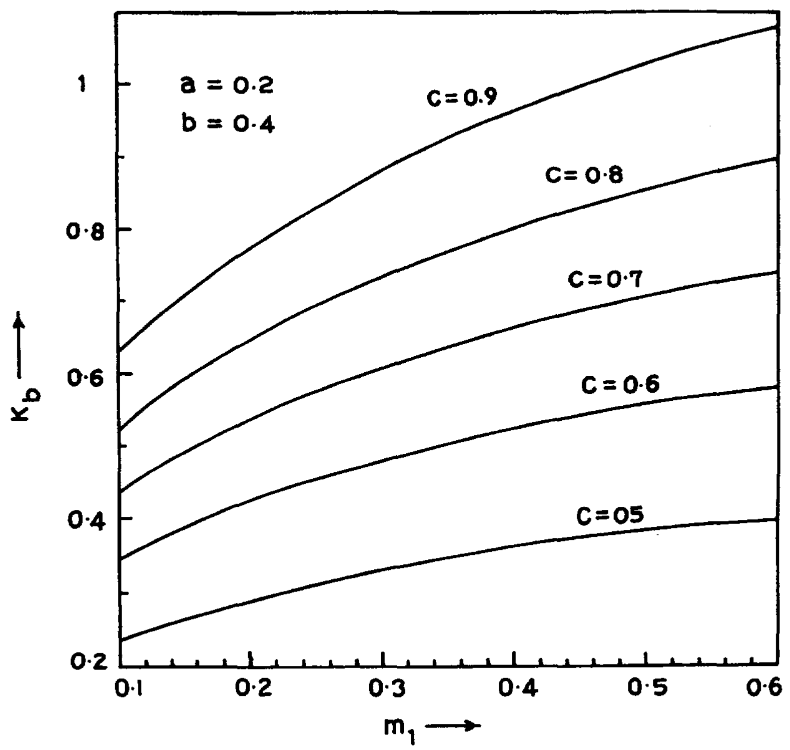

Fig. 11. Stress intensity factor K_b vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

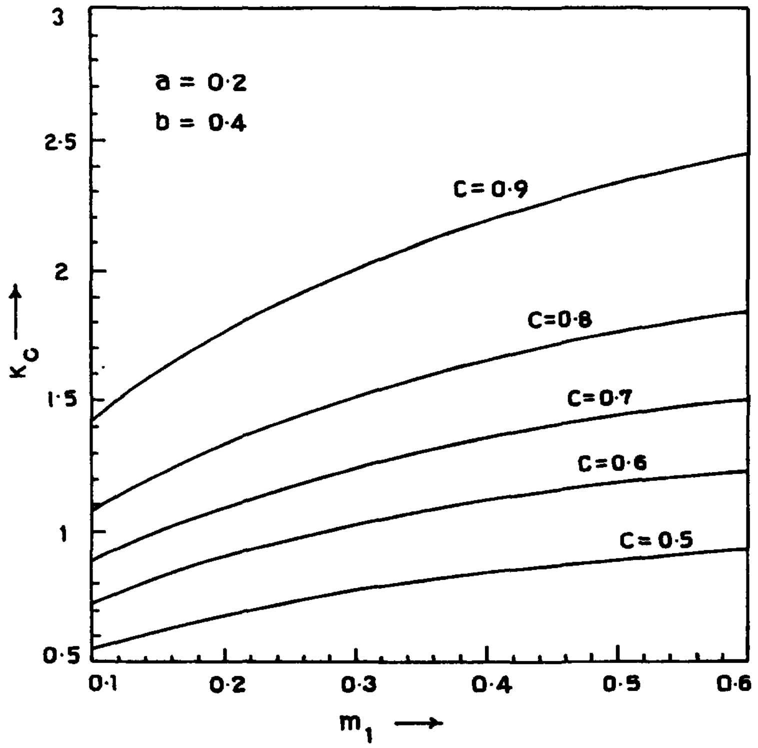

Fig. 12. Stress intensity factor K_c vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

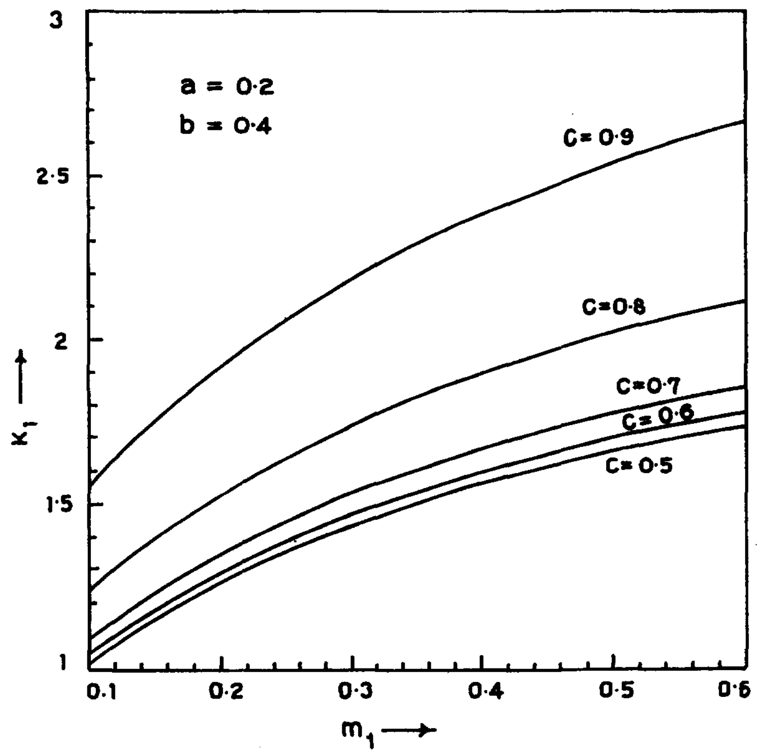

Fig. 13. Stress intensity factor K_1 vs dimensionless frequency m_1 for $a = 0.2$, $b = 0.4$ and for different values of c.

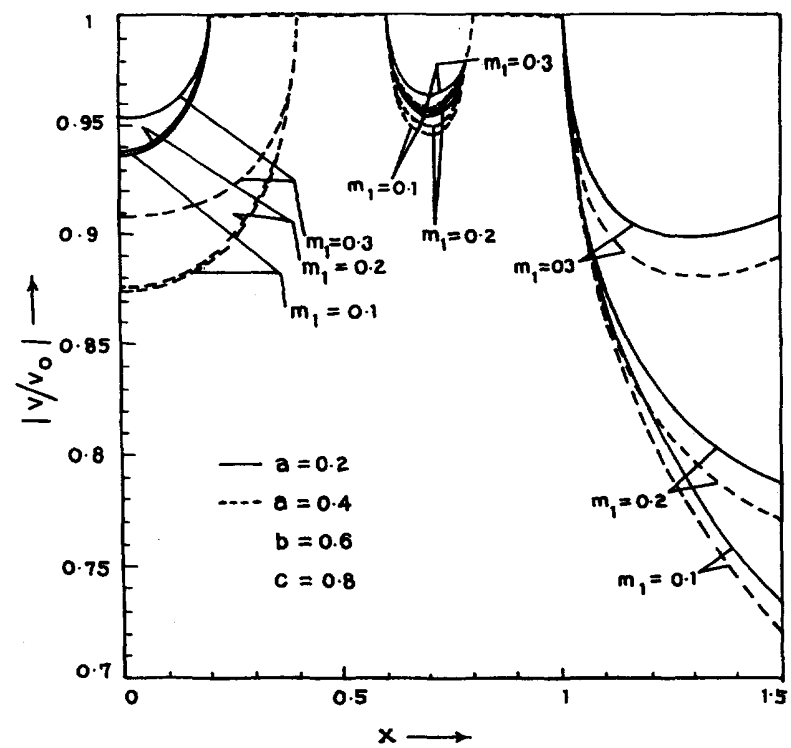

Fig. 14. Vertical displacement $|v(x,0)/v_0|$ vs dimensionless distance x for $b = 0.6$, $c = 0.8$, $a = 0.2$, 0.4 and for $m_1 = 0.1, 0.2, 0.3$.

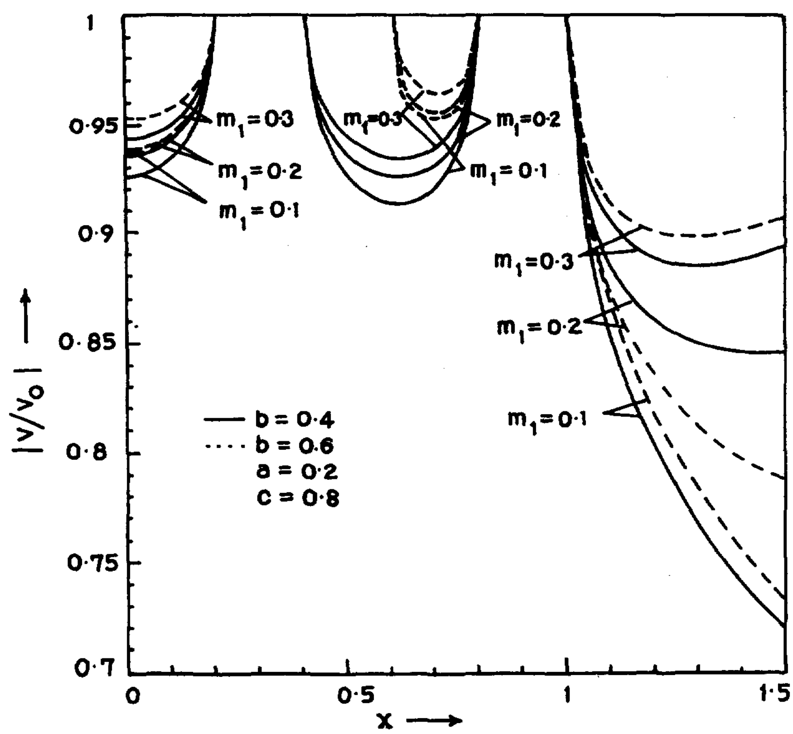

Fig. 15. Vertical displacement $|v(x,0)/v_0|$ vs dimensionless distance x for $a = 0.2$, $c = 0.8$, $b = 0.4$, 0.6 and for $m_1 = 0.1, 0.2, 0.3$.

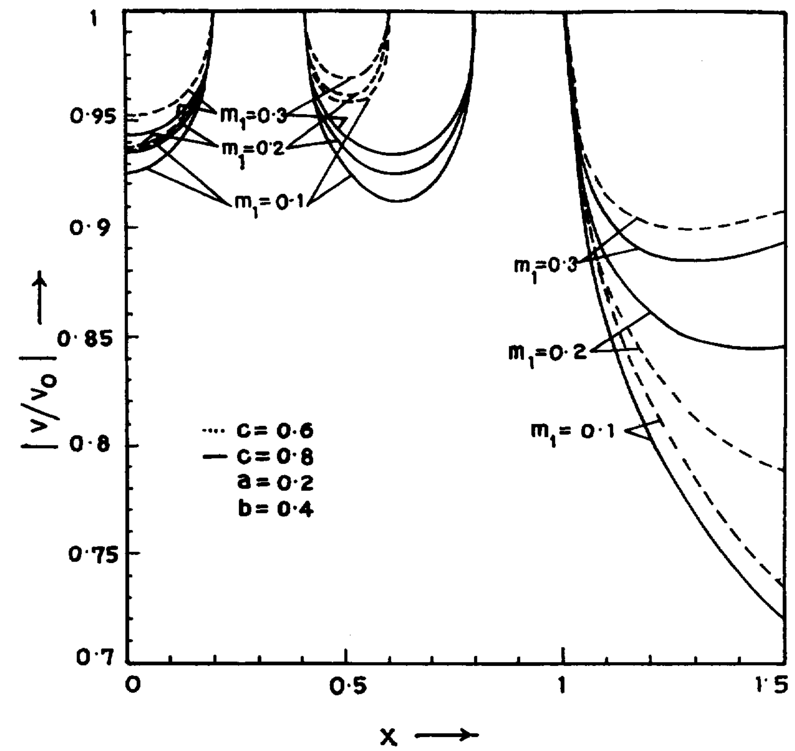

Fig. 16. Vertical displacement $|v(x, 0)v_0|$ vs dimensionless distance x for $a = 0.2$, $b = 0.4$, $c = 0.6$, 0.8 and for $m_1 = 0.1, 0.2, 0.3$.

The vertical displacement has been plotted for different strip lengths. It is found from Figs 14–16 that with an increase in value of strip lengths, the displacement increases.

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